21–499 Report, Kartik Sabharwal (ksabharw)
`https://github.com/Fyrbll/CombiGames`

Bowling Description
Two people — we'll call them A and B — are playing a bowling game.
They have n bowling pins in a row, and they take turns to knock over pins using special bowling balls until there are no pins left standing.
The first bowling ball is *small*, and it always knocks over just one pin.
The other one is *large*, and it always knocks over exactly 3 pins. To be clear, a player is only allowed to use a large ball on three pins that are right next to each other.
No player ever misses.
A goes first, and if a player has no available moves in her turn (i.e. if the player that went before knocked over all the pins) then she loses.

Bowling Analysis
Our goal is to find the nimber for any n-size row of starting pins.
Let the function mapping n to its nimber be $\mathcal{N}$.
Also, number the pins 1, 2, . . ., n from left to right.

Claim:

$$\mathcal{N}(n) = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Proof:
We will prove this by strong induction.

Base Cases:
$\mathcal{N}(0) = 0$
$\mathcal{N}(1) = 1$
$\mathcal{N}(2) = 0$
$\mathcal{N}(3) = 1$
Thus, the claim holds for $n = 0, 1, 2$ or 3.

Induction Hypothesis:
Suppose for some $m \geq 4$, the claim is true for all $i < m$.

Induction Step:
We want to prove that the claim is true for $m + 1$.

If $m + 1$ is odd, then throwing a large bowling ball will reduce the number of pins to $m - 2$, and throwing a small bowling ball will reduce the number of pins to m. Notice that both m and $m - 2$ must be even numbers. To sum up, throwing any bowling ball will give us an even number of pins.
If the pins knocked over are all on either the left or right end of the row, then we will be left with a continuous row of an even number of pins. By the I.H. such a position has nimber 0.
If the pins knocked over are somewhere in the middle of the row, we will be left with two continuous rows of pins, with a "gap" in the middle that was formerly occupied by the knocked pins. These two continuous rows must either both have an even number of pins or both have an odd number of pins. In the former case, the situation will have nimber $0 \oplus 0 = 0$ and in the latter case, the situation will have nimber $1 \oplus 1 = 0$.
It follows that all the "child" positions of the situation with a row of $m + 1$ pins have nimber 0. So, the minimum excluded value is 1, which means $\mathcal{N}(m + 1) = 1$ when $m + 1$ is odd.

If $m + 1$ is even, then throwing a large bowling ball will reduce the number of pins to $m - 2$, and throwing a small bowling ball will reduce the number of pins to m. Notice that both m and $m - 2$ must now be odd numbers. To sum up, throwing any bowling ball will give us an odd number of pins.
If the pins knocked over are all on either the left or right end of the row, then we will be left with a continuous row of an odd number of pins. By the I.H. such a position has nimber 1.
If the pins knocked over are somewhere in the middle of the row, we will be left with two continuous rows of pins, with a "gap" in the middle that was formerly occupied by the knocked pins. One of these rows must have an even number of pins and the other row must have an odd number of pins. By the I.H. in this situation the nimber is either $0 \oplus 1 = 1$ or $1 \oplus 0 = 1$.
It follows that all the "child" positions of the situation with a row of $m + 1$ pins have nimber 1. So, the minimum excluded value is 0, which means $\mathcal{N}(m + 1) = 0$ when $m + 1$ is even.

We have proved the claim to be true for $m + 1$.
Thus by strong induction the claim must hold for all natural numbers. ■

Bowling Possibilities

Suppose the rule about the use of the *large* bowling ball is altered, and now players can disregard "gaps" in the row of pins while throwing the large ball.
To illustrate, in the previous game if a player was faced with the following situation (each vertical line represents a pin, and each star represents a position in the row previously occupied by a pin but now empty):

```
| | * | | |
```

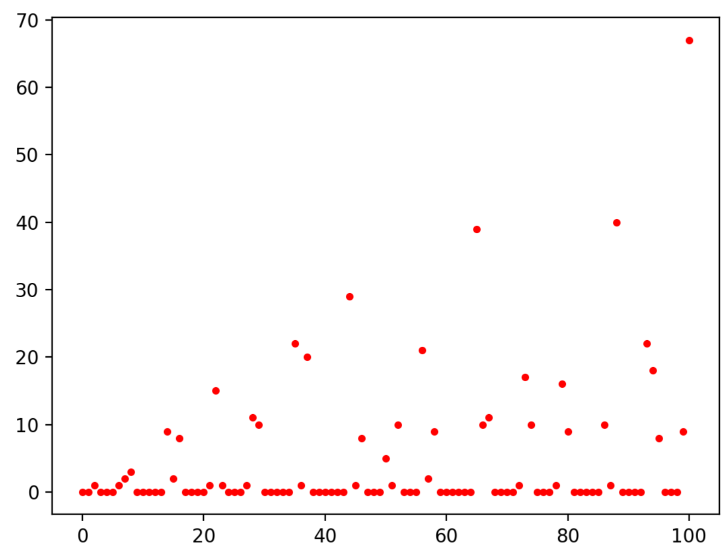
then the players would have to treat it as two "disjoint" piles.
Now, they are allowed to throw a large ball in such a way that they can turn the above position to the following one in a single move:

```
| * * * | |
```

One might ask how this changes the nimbers that can be achieved. The answer is that it makes things much less well-behaved. The nimbers from $n = 0$ to $n = 100$, in array form, are:

```
rawNimbers = [ 0, 0, 1, 0, 0, 0, 1, 2, 3, 0,
               0, 0, 0, 0, 9, 2, 8, 0, 0, 0,
               0, 1,15, 1, 0, 0, 0, 1,11,10,
               0, 0, 0, 0, 0,22, 1,20, 0, 0,
               0, 0, 0, 0,29, 1, 8, 0, 0, 0,
               5, 1,10, 0, 0, 0,21, 2, 9, 0,
               0, 0, 0, 0, 0,39,10,11, 0, 0,
               0, 0, 1,17,10, 0, 0, 0, 1,16,
               9, 0, 0, 0, 0, 0,10, 1,40, 0,
               0, 0, 0,22,18, 8, 0, 0, 0, 9,
               67 ]
```

We can plot with n on the x-axis versus $\mathcal{N}(n)$ on the y-axis to get:



I looked at the nimbers, as well as the differences between these nimbers, and there were no good, reliable patterns. I tried to prove first that there was no upper bound on the nimbers as n increases, and also that there are infinitely many 0-positions, but I could not get that to work.
Furthermore, I believe that more data is necessary to see if this eventually becomes periodic or not, but even after spending a considerable amount of time memoizing and optimizing my code, computing nimbers for n higher than 100 did not take viable amounts of time.
I moved to a new game, which I will discuss in the next section.

Transfer Description
This is a game for two players, who take turns making moves.
We have three points $\mathbb{P} = \{P_1, P_2, P_3\}$ in a plane.
Consider a function $s : \mathbb{P} \to \mathbb{N}$, where $s(P_i)$ denotes the number of "stones" at point P_i.
Initially, $s(P_1) = s(P_3) = 0$ and $s(P_2) = n$.
A player, in her turn, may either move some stones from P_2 to P_1 or she may move some stones from P_2 to P_3, but she can't do both.
Also, the following invariants must be respected:

1. $s(P_1) \leq s(P_2)$ and $s(P_3) \leq s(P_2)$.
2. If, at the start of a player's turn, $s(P_1) < s(P_3)$ then at the end of the turn $s(P_1) \leq s(P_3)$ must hold.
   Similarly, if at the start of a player's turn it is true that $s(P_3) < s(P_1)$ then at the end of the turn $s(P_3) \leq s(P_1)$ must hold.

If a player cannot make any moves, then she loses.
This means we can describe a game of this form with a tuple $(a, \quad b, \quad c)$ which means $s(P_1) = a, s(P_2) = b, s(P_3) = c$.

Transfer Analysis
We claim that:

$$\mathcal{N}(0, \quad n, \quad 0) = \begin{cases} 0, & \text{if } n \text{ is odd or} n = 0 \\ 1, & \text{if } n \text{ is even and } n \neq 4 \\ 2, & \text{if } n = 4 \end{cases}$$

We can easily find out that:
$\mathcal{N}(0, \quad 0, \quad 0) \ = 0$
$\mathcal{N}(0, \quad 1, \quad 0) \ = 1$
$\mathcal{N}(0, \quad 2, \quad 0) \ = 0$
$\mathcal{N}(0, \quad 3, \quad 0) \ = 1$
$\mathcal{N}(0, \quad 4, \quad 0) \ = 2$
$\mathcal{N}(0, \quad 5, \quad 0) \ = 1$

Then, we prove all the following for $n \geq 0$:
$\mathcal{N}(0, \quad 6n+6, \quad 0) \ = 0$
$\mathcal{N}(0, \quad 6n+7, \quad 0) \ = 1$
$\mathcal{N}(0, \quad 6n+8, \quad 0) \ = 0$
$\mathcal{N}(0, \quad 6n+9, \quad 0) \ = 1$
$\mathcal{N}(0, \quad 6n+10, \quad 0) \ = 0$
$\mathcal{N}(0, \quad 6n+11, \quad 0) \ = 1$

Each of the proofs involves casing on whether the number of stones moved by the first player is $< \frac{6n+i}{3}$ and even, $< \frac{6n+i}{3}$ and odd, or just $\geq \frac{6n+i}{3}$.

Note that we provide a full induction proof for the $(0, \quad 6n+8, \quad 0)$ case, and for the others we show exactly how to modify the first proof to give us the needed result.

Proof for $6n + 8$ Case
We want to prove the following proposition:

$$P(n) : (0, \quad 6n+8, \quad 0) \;\; \cong \; *1 \;\text{ where }\; n \geq 0$$

We will use strong induction to prove the above statement.

Base Cases:
Using our program, we have verified that $P(n)$ holds for $n \in \{0, 1, 2, 3\}$.
The fact that $(0, \quad 2, \quad 0) \;\; \cong \; *1$ will also be useful.

Induction Hypothesis:
Assume for some $k \geq 1$ that $\forall\, n < k,\; P(n)$ holds.

Induction Step:
We need to prove that $P(k)$ holds. In other words, we need to show:

$$(0, \quad 6k+8, \quad 0) \;\; \cong \; *1$$

Simplifying the problem,
To show that $(0, \quad 6k+8, \quad 0)$ has nimber 1, it suffices to show the following:

1. In one move, we can reach a state with nimber 0
2. In one move, we cannot reach a state with nimber 1

Looking at the initial state's children,
Notice that in one move from the original state we can reach any state of the form:

$$(m, \quad 6k+8-m, \quad 0) \;\text{ where } 1 \leq m \leq 3k+4$$

the remaining states that we can reach from the original state are of the form:

$$(0, \quad 6k+8-m, \quad m) \;\text{ where } 1 \leq m \leq 3k+4$$

Since the latter form is symmetric to the former, focusing our analysis on the former is enough to complete the proof.

When $m = 3k+4$:
Observe that when $m = 3k+4$, we obtain the state $(3k+4, \quad 3k+4, \quad 0)$ which clearly has nimber 0, so we have proved the first fact we need.

When m is odd and $1 \leq m \leq 2k+2$:
When m is an odd number satisfying $1 \leq m \leq 2k+2$, notice that

$$(m+1, \quad 3k+4-\tfrac{m+1}{2}, \quad 3k+4-\tfrac{m+1}{2}) \;\; \cong \; *0$$

Work one step backwards from this to get

$$(m, \quad 3k+5-\tfrac{m+1}{2}, \quad 3k+4-\tfrac{m+1}{2}) \;\; \cong \; *1$$

Observe that

$$(m, \quad 6k+8-m, \quad 0) \;\xrightarrow{\text{one move}}\; (m, \quad 3k+5-\tfrac{m+1}{2}, \quad 3k+4-\tfrac{m+1}{2})$$

It follows that the minimum excluded value of the nimbers of all the child states of $(m, \quad 6k+8-m, \quad 0)$ cannot be 1, and thus the nimber of this state cannot be 1, by the Sprague-Grundy theorem.

When m is even and $1 \leq m \leq 2k+2$:
When m is an even number satisfying $1 \leq m \leq 2k+2$, observe that

$$(m, \quad 6k+8-m, \quad 0) \xrightarrow{\text{one move}} (m, \quad 6k+8-2m, \quad m)$$

We can replace m with $2i$ where $1 \leq i \leq k+1$ to get

$$(2i, \quad 6k+8-2i, \quad 0) \xrightarrow{\text{one move}} (2i, \quad 6k+8-4i, \quad 2i)$$

In the resultant state, the difference between the left and middle values, also the difference between the right and middle values, is

$$6(k-i)+8$$

Which intuitively suggests that

$$(2i, \quad 6k+8-4i, \quad 2i) \;\cong\; (0, \quad 6(k-i)+8, \quad 0) \;\; \text{for } 1 \leq i \leq k+1$$

and although we lack a rigorous argument for this we will assume it is true.
Our knowledge that $(0, \quad 2, \quad 0) \;\cong\; *1$ and also our induction hypothesis, tell us:

$$(0, \quad 6(k-i)+8, \quad 0) \;\cong\; *1 \text{ for } 1 \leq i \leq k+1$$

Consequently, $(2i, \quad 6k+8-4i, \quad 2i) \;\cong\; *1$.
It follows that the minimum excluded value of the nimbers of all the child states of $(m, \quad 6k+8-m, \quad 0)$ cannot be 1, and thus the nimber of this state cannot be 1.

When $m \geq 2k+3$:
If $m \geq 2k+3$, it is clear that

$$(m, \quad m, \quad 6k+8-2m) \;\cong\; *0$$

Working backwards from this, we get that

$$(m, \quad m+1, \quad 6k+7-2m) \;\cong\; *1$$

We know that

$$(m, \quad 6k+8-m, \quad 0) \xrightarrow{\text{one move}} (m, \quad m+1, \quad 6k+7-2m)$$

It follows that the minimum excluded value of the nimbers of all the child states of $(m, \quad 6k+8-m, \quad 0)$ cannot be 1, and thus the nimber of this state cannot be 1.

Conclusion:
1 must be the mex of the nimbers of the children of $(0, \quad 6k+8, \quad 0)$, so we have proved that $(0, \quad 6k+8, \quad 0) \;\cong\; *1$, using the Sprague-Grundy theorem.

Details for $6n+6$ Case
Consider the initial position

$$(0, \quad 6n+6, \quad 0) \text{ where } n \geq 0$$

The first move will take the initial position to a position of the form

$$(m, \quad 6n+6-m, \quad 0) \text{ where } 1 \leq m \leq 3n+3$$

If $m = 3n+3$:
This is the state $(3n+3, \quad 3n+3, \quad 0)$ which clearly has nimber 0.

If $m \leq 2n+2$ and m is odd:
Then, consider the following move sequence,

$$\begin{aligned} & (m, \quad 6n+6-m, \quad 0) \\ \xrightarrow{\text{one possible child}} & \left(m, \quad 3n+4-\tfrac{m+1}{2}, \quad 3n+3-\tfrac{m+1}{2}\right) \\ \xrightarrow{\text{only child}} & \left(m+1, \quad 3n+3-\tfrac{m+1}{2}, \quad 3n+3-\tfrac{m+1}{2}\right) \end{aligned}$$

If $m \leq 2n+2$ and m is even:
Consider the following move sequence:

$$\begin{aligned} & (m, \quad 6n+6-m, \quad 0) \\ \xrightarrow{\text{one possible child}} & (m, \quad 6n+6-2m, \quad m) \\ \cong & (0, \quad 6n+6-3m, \quad 0) \end{aligned}$$

Since m is even, we can write it as $2i$ for some $1 \leq i \leq n+1$.

$$\begin{aligned} & (0, \quad 6n+6-3m, \quad 0) \\ \cong & (0, \quad 6n+6-6i, \quad 0) \\ \cong & (0, \quad 6(n-i)+6, \quad 0) \end{aligned}$$

By our induction hypothesis, $(0, \quad 6(n-i)+6, \quad 0)$ has nimber 1.

If $m \geq 2n+3$:
The move sequence below works:

$$\begin{aligned} & (m, \quad 6n+6-m, \quad 0) \\ \xrightarrow{\text{one possible child}} & (m, \quad m+1, \quad 6n+5-2m) \\ \xrightarrow{\text{only child}} & (m, \quad m, \quad 6n+6-2m) \end{aligned}$$

Which provides us with everything we need to fill in the proof skeleton.

Details for $6n+10$ Case
Consider the initial position

$$(0, \quad 6n+10, \quad 0) \quad \text{where } n \geq 0$$

The first move will take the initial position to a position of the form

$$(m, \quad 6n+6-m, \quad 0) \quad \text{where } 1 \leq m \leq 3n+3$$

<u>If $m = 3n+5$:</u>
This is the state $(3n+5, \quad 3n+5, \quad 0)$ which clearly has nimber 0.

<u>If $m \leq 2n+3$ and m is odd:</u>
Then, consider the following move sequence,

$$\begin{aligned} & (m, \quad 6n+10-m, \quad 0) \\ \xrightarrow{\text{one possible child}} & (m, \quad 3n+6-\tfrac{m+1}{2}, \quad 3n+5-\tfrac{m+1}{2}) \\ \xrightarrow{\text{only child}} & (m+1, \quad 3n+5-\tfrac{m+1}{2}, \quad 3n+5-\tfrac{m+1}{2}) \end{aligned}$$

<u>If $m \leq 2n+3$ and m is even:</u>
Consider the following move sequence:

$$\begin{aligned} & (m, \quad 6n+10-m, \quad 0) \\ \xrightarrow{\text{one possible child}} & (m, \quad 6n+10-2m, \quad m) \\ \cong & (0, \quad 6n+10-3m, \quad 0) \end{aligned}$$

Since m is even, we can write it as $2i$ for some $1 \leq i \leq n+1$.

$$\begin{aligned} & (0, \quad 6n+10-3m, \quad 0) \\ \cong & (0, \quad 6n+10-6i, \quad 0) \\ \cong & (0, \quad 6(n-i)+10, \quad 0) \end{aligned}$$

By our induction hypothesis, $(0, \quad 6(n-i)+10, \quad 0)$ has nimber 1.

<u>If $m \geq 2n+4$:</u>
The move sequence below works:

$$\begin{aligned} & (m, \quad 6n+10-m, \quad 0) \\ \xrightarrow{\text{one possible child}} & (m, \quad m+1, \quad 6n+9-2m) \\ \xrightarrow{\text{only child}} & (m, \quad m, \quad 6n+10-2m) \end{aligned}$$

Which provides us with everything we need to fill in the proof skeleton.

Details for $6n + 7$ Case

In these three situations our goal is different — we need to show that the root position has the nimber 0. For this, we should show that all the "children" of this position have nimber > 0. Now to show that a position has nimber > 0, we need to show that it in turn has at least one child with nimber 0. Using case analysis, we accomplish that goal.

Consider the initial position

$$(0, \quad 6n + 7, \quad 0) \quad \text{where } n \geq 0$$

The first move brings $(0, \quad 6n + 7, \quad 0)$ to a state of the form $(m, \quad 6n + 7 - m, \quad 0)$, where $1 \leq m \leq 3n + 3$.
If $m = 3n + 3$,

$$\begin{aligned} & (3n + 3, \quad 3n + 4, \quad 0) \\ \xrightarrow{\text{only child}} \; & (3n + 3, \quad 3n + 3, \quad 1) \end{aligned}$$

<u>If $m \leq 2n + 2$ and m is odd:</u>
Then, consider the following move sequence,

$$\begin{aligned} & (m, \quad 6n + 7 - m, \quad 0) \\ \xrightarrow{\text{one possible child}} \; & (m, \quad \tfrac{6n+7-m}{2}, \quad \tfrac{6n+7-m}{2}) \end{aligned}$$

<u>If $m \leq 2n + 2$ and m is even:</u>
Consider the following move sequence:

$$\begin{aligned} & (m, \quad 6n + 7 - m, \quad 0) \\ \xrightarrow{\text{one possible child}} \; & (m, \quad 6n + 7 - 2m, \quad m) \\ \cong \; & (0, \quad 6n + 7 - 3m, \quad 0) \end{aligned}$$

Since m is odd, we can write it as $2i$ for some $1 \leq i \leq n + 1$.

$$\begin{aligned} & (0, \quad 6n + 7 - 3m, \quad 0) \\ \cong \; & (0, \quad 6n + 7 - 6i, \quad 0) \\ \cong \; & (0, \quad 6(n - i) + 7, \quad 0) \end{aligned}$$

By our induction hypothesis, $(0, \quad 6(n - i) + 7, \quad 0)$ has nimber 0.

<u>If $m \geq 2n + 3$:</u>
The move sequence below works:

$$\begin{aligned} & (m, \quad 6n + 7 - m, \quad 0) \\ \xrightarrow{\text{one possible child}} \; & (m, \quad m, \quad 6n + 7 - 2m) \end{aligned}$$

Details for $6n+9$ Case
Consider the initial position

$$(0, \quad 6n+9, \quad 0) \quad \text{where } n \geq 0$$

The first move brings $(0, \quad 6n+9, \quad 0)$ to a state of the form $(m, \quad 6n+9-m, \quad 0)$, where $1 \leq m \leq 3n+4$.
If $m = 3n+4$,

$$\begin{aligned} & (3n+4, \quad 3n+5, \quad 0) \\ \xrightarrow{\text{only child}} & (3n+4, \quad 3n+4, \quad 1) \end{aligned}$$

<u>If $m \leq 2n+3$ and m is odd:</u>
Then, consider the following move sequence,

$$\begin{aligned} & (m, \quad 6n+9-m, \quad 0) \\ \xrightarrow{\text{one possible child}} & (m, \quad \tfrac{6n+9-m}{2}, \quad \tfrac{6n+9-m}{2}) \end{aligned}$$

<u>If $m \leq 2n+3$ and m is even:</u>
Consider the following move sequence:

$$\begin{aligned} & (m, \quad 6n+9-m, \quad 0) \\ \xrightarrow{\text{one possible child}} & (m, \quad 6n+9-2m, \quad m) \\ \cong & (0, \quad 6n+9-3m, \quad 0) \end{aligned}$$

Since m is odd, we can write it as $2i$ for some $1 \leq i \leq n+1$.

$$\begin{aligned} & (0, \quad 6n+9-3m, \quad 0) \\ \cong & (0, \quad 6n+9-6i, \quad 0) \\ \cong & (0, \quad 6(n-i)+9, \quad 0) \end{aligned}$$

By our induction hypothesis, $(0, \quad 6(n-i)+9, \quad 0)$ has nimber 0.

<u>If $m \geq 2n+3$:</u>
The move sequence below works:

$$\begin{aligned} & (m, \quad 6n+9-m, \quad 0) \\ \xrightarrow{\text{one possible child}} & (m, \quad m, \quad 6n+9-2m) \end{aligned}$$

Details for $6n + 11$ Case

Consider the initial position

$$(0,\quad 6n+11,\quad 0)\quad \text{where } n \geq 0$$

The first move brings $(0,\quad 6n+11,\quad 0)$ to a state of the form $(m,\quad 6n+11-m,\quad 0)$, where $1 \leq m \leq 3n+5$.
If $m = 3n+5$,

$$\begin{aligned} &(3n+5,\quad 3n+6,\quad 0) \\ \xrightarrow{\text{only child}}\ &(3n+5,\quad 3n+5,\quad 1) \end{aligned}$$

If $m \leq 2n+3$ and m is odd:

Then, consider the following move sequence,

$$\begin{aligned} &(m,\quad 6n+11-m,\quad 0) \\ \xrightarrow{\text{one possible child}}\ &(m,\quad \tfrac{6n+11-m}{2},\quad \tfrac{6n+11-m}{2}) \end{aligned}$$

If $m \leq 2n+3$ and m is even:

Consider the following move sequence:

$$\begin{aligned} &(m,\quad 6n+11-m,\quad 0) \\ \xrightarrow{\text{one possible child}}\ &(m,\quad 6n+11-2m,\quad m) \\ \cong\ &(0,\quad 6n+11-3m,\quad 0) \end{aligned}$$

Since m is odd, we can write it as $2i$ for some $1 \leq i \leq n+1$.

$$\begin{aligned} &(0,\quad 6n+11-3m,\quad 0) \\ \cong\ &(0,\quad 6n+11-6i,\quad 0) \\ \cong\ &(0,\quad 6(n-i)+11,\quad 0) \end{aligned}$$

By our induction hypothesis, $(0,\quad 6(n-i)+11,\quad 0)$ has nimber 0.

If $m \geq 2n+3$:

The move sequence below works:

$$\begin{aligned} &(m,\quad 6n+11-m,\quad 0) \\ \xrightarrow{\text{one possible child}}\ &(m,\quad m,\quad 6n+11-2m) \end{aligned}$$

Transfer Possibilities

Consider a variation on the game described above, where we have the same point set $\mathbb{P}$ as before, and in addition to what a player could do before, it is also possible for a player to move stones from P_1 to P_3 or vice versa, provided the following invariants are respected:

1. $s(P_1) \leq s(P_2)$ and $s(P_3) \leq s(P_2)$.
2. If the first move makes $s(P_1) < s(P_3)$, then for the rest of the game $s(P_1) \leq s(P_3)$ must hold.
   If the first move makes $s(P_3) < s(P_1)$, then for the rest of the game $s(P_3) \leq s(P_1)$ must hold.

Also notice that if we have a game state of the form

$$(a, \quad b, \quad c)$$

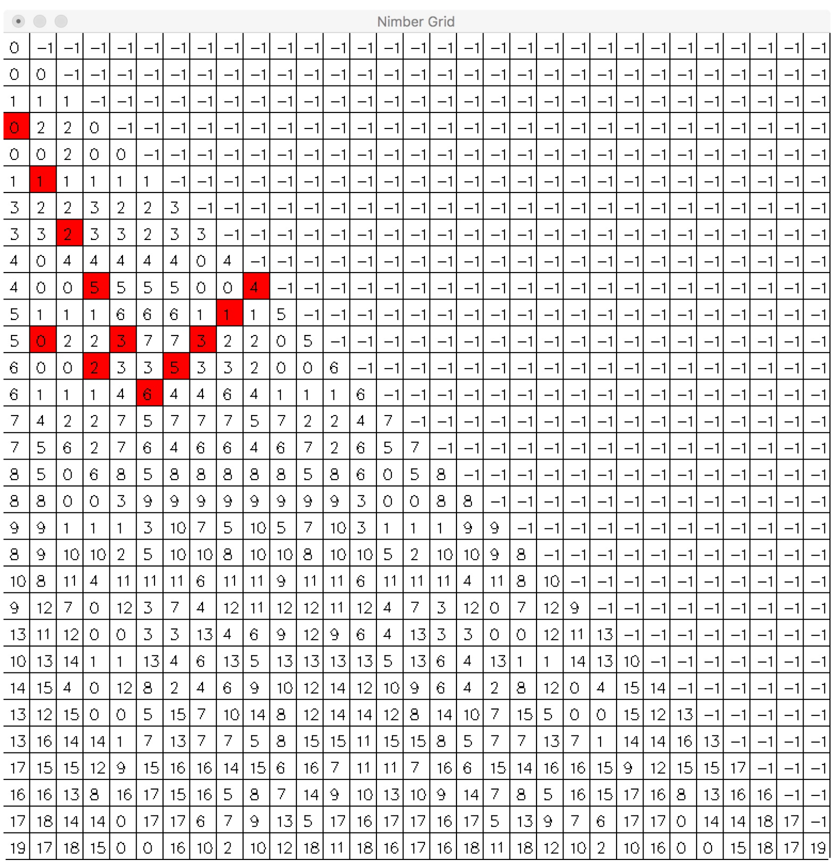
we can assume WLOG that $a \geq c$, and according to the invariant we know $b \geq a$. Then we can say

$$(a, \quad b, \quad c) \;\cong (a - c, \quad b - c, \quad 0)$$

and this knowledge allows us to represent any state as just a 2-tuple.

Now I worked on analyzing this situation for a while without any results, because the numbers seemed quite random. Then when another student in the class started to work on this problem, he pointed out that his data was different, prompting me to hunt for bugs in my code. I have since corrected the bugs to the best of my ability, but since I had moved on to a different problem, I was not able to return to exploring this.

In the `Triangle` folder among my code for this game, I have included a program called `interactive` that when run with a numerical command line argument n (e.g. `./interactive 31`) produces an $n \times n$ grid with cells where the cell (r, c) (both row and column are zero-indexed) contains the nimber for the game $(c, \quad r, \quad 0)$. If a cell has the number -1 in it, it is an invalid state and should be ignored. Note that clicking on a cell highlights all the cells representing its child states, making it easier for a user to investigate what is really going on in the game. Actually, the way the squares are highlighted in the diagram below inspired the creation of the next game that we are about to consider. It arose as a simplification of this game's pattern.

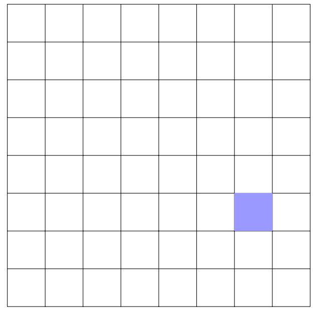


Nimber Grid

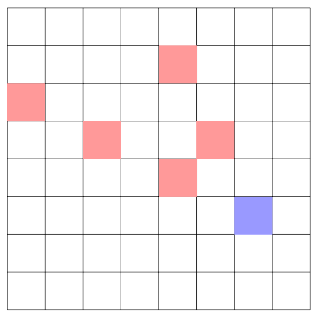
0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
0	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
0	2	2	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
0	0	2	0	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
3	2	2	3	2	2	3	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
3	3	2	3	3	2	3	3	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
4	0	4	4	4	4	4	0	4	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
4	0	0	5	5	5	5	0	0	4	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
5	1	1	1	6	6	6	1	1	1	5	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
5	0	2	2	3	7	7	3	2	2	0	5	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
6	0	0	2	3	3	5	3	3	2	0	0	6	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
6	1	1	1	4	6	4	4	6	4	1	1	1	6	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
7	4	2	2	7	5	7	7	7	5	7	2	2	4	7	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
7	5	6	2	7	6	4	6	6	4	6	7	2	6	5	7	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
8	5	0	6	8	5	8	8	8	8	8	5	8	6	0	5	8	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
8	8	0	0	3	9	9	9	9	9	9	9	9	3	0	0	8	8	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
9	9	1	1	1	3	10	7	5	10	5	7	10	3	1	1	1	9	9	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
8	9	10	10	2	5	10	10	8	10	10	8	10	10	5	2	10	10	9	8	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
10	8	11	4	11	11	11	6	11	11	9	11	11	6	11	11	11	4	11	8	10	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1
9	12	7	0	12	3	7	4	12	11	12	12	11	12	4	7	3	12	0	7	12	9	−1	−1	−1	−1	−1	−1	−1	−1	−1
13	11	12	0	0	3	3	13	4	6	9	12	9	6	4	13	3	3	0	0	12	11	13	−1	−1	−1	−1	−1	−1	−1	−1
10	13	14	1	1	13	4	6	13	5	13	13	13	13	5	13	6	4	13	1	1	14	13	10	−1	−1	−1	−1	−1	−1	−1
14	15	4	0	12	8	2	4	6	9	10	12	14	12	10	9	6	4	2	8	12	0	4	15	14	−1	−1	−1	−1	−1	−1
13	12	15	0	0	5	15	7	10	14	8	12	14	14	12	8	14	10	7	15	5	0	0	15	12	13	−1	−1	−1	−1	−1
13	16	14	14	1	7	13	7	7	5	8	15	15	11	15	15	8	5	7	7	13	7	1	14	14	16	13	−1	−1	−1	−1
17	15	15	12	9	15	16	16	14	15	6	16	7	11	11	7	16	6	15	14	16	16	15	9	12	15	15	17	−1	−1	−1
16	16	13	8	16	17	15	16	5	8	7	14	9	10	13	10	9	14	7	8	5	16	15	17	16	8	13	16	16	−1	−1
17	18	14	14	0	17	17	6	7	9	13	5	17	16	17	17	16	17	5	13	9	7	6	17	17	0	14	14	18	17	−1
19	17	18	15	0	0	16	10	2	10	12	18	11	18	16	17	16	18	11	18	12	10	2	10	16	0	0	15	18	17	19

Chess Description

Suppose we have a chessboard, and there's a knight somewhere on this chessboard. The location of the knight is indicated by a blue square. As an example consider the drawing below:



Two people look at this board and decide to play a game.
One tells the other "Let's take turns moving this knight towards the top left of this board using only knight-like moves. If I am left with no legal moves on my turn, I lose. If you can't make any move on your turn, you lose", and since she is bored, she readily agrees.
What can we say about this game? For clarity, the moves that the first player can make are highlighted in red below. The the possible moves for the rest of the game should be clear . . .



Note that the term "single-up double-left move" means a move where the knight moves i squares above and $2i$ squares to the left. What this i will be understood easily from the context of the explanation.
Similarly the term "double-up single-left move" means a move where the knight moves $2i$ squares above and i squares to the left. What this i will be understood easily from the context of the explanation.
Observe that every square has a certain nimber, so *ideally* we should come up with a system to figure out the nimber given a row, column pair.
For additional inspiration/motivation, a manually computed and color-coded diagram of the board is presented below:

Chessboard With Nimbers

-	A	B	C	D	E	F	G	H	I
A	0	0	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1	1	1
C	0	1	1	1	2	2	2	2	2
D	0	1	1	0	2	2	3	3	3
E	0	1	2	2	2	3	3	3	4
F	0	1	2	2	3	0	3	2	4
G	0	1	2	3	3	3	1	4	4
H	0	1	2	3	3	2	4	0	4
I	0	1	2	3	4	4	4	4	1

This is a larger, uncolored version of the board with more nimbers. There are 26 rows and 26 colums.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
0	1	1	0	2	2	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
0	1	2	2	2	3	3	3	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
0	1	2	2	3	0	3	2	4	4	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
0	1	2	3	3	3	1	4	4	5	5	5	6	6	6	6	6	6	6	6	6	6	6	6	6	6
0	1	2	3	3	2	4	0	4	5	5	4	6	6	7	7	7	7	7	7	7	7	7	7	7	7
0	1	2	3	4	4	4	4	1	5	6	3	6	7	7	7	8	8	8	8	8	8	8	8	8	8
0	1	2	3	4	4	5	5	5	0	6	6	7	7	7	6	8	8	9	9	9	9	9	9	9	9
0	1	2	3	4	5	5	5	6	6	1	7	7	7	5	8	8	9	9	9	10	10	10	10	10	10
0	1	2	3	4	5	5	4	3	6	7	0	7	6	8	4	8	6	9	8	10	10	11	11	11	11
0	1	2	3	4	5	6	6	6	7	7	7	1	8	8	8	5	9	10	10	10	11	11	11	12	12
0	1	2	3	4	5	6	6	7	7	7	6	8	0	8	2	9	9	9	10	11	11	11	10	12	12
0	1	2	3	4	5	6	7	7	7	5	8	8	8	1	9	3	9	10	5	11	11	9	9	12	13
0	1	2	3	4	5	6	7	7	6	8	4	8	2	9	0	9	2	10	10	6	10	12	8	12	13
0	1	2	3	4	5	6	7	8	8	8	8	5	9	3	9	1	9	10	11	11	7	12	12	13	13
0	1	2	3	4	5	6	7	8	8	9	6	9	9	9	2	9	0	10	2	11	12	12	13	13	13
0	1	2	3	4	5	6	7	8	9	9	9	10	9	10	10	10	10	1	11	3	12	12	13	13	12
0	1	2	3	4	5	6	7	8	9	9	8	10	10	5	10	11	2	11	0	11	4	11	13	6	14
0	1	2	3	4	5	6	7	8	9	10	10	10	11	11	6	11	11	3	11	1	12	13	13	14	14
0	1	2	3	4	5	6	7	8	9	10	10	11	11	11	10	7	12	12	4	12	0	12	13	14	14
0	1	2	3	4	5	6	7	8	9	10	11	11	11	9	12	12	12	12	11	13	12	1	14	3	15
0	1	2	3	4	5	6	7	8	9	10	11	11	10	9	8	12	13	13	13	13	13	14	0	14	4
0	1	2	3	4	5	6	7	8	9	10	11	12	12	12	12	13	13	13	6	14	14	3	14	1	14
0	1	2	3	4	5	6	7	8	9	10	11	12	12	13	13	13	13	12	14	14	14	15	4	14	0

Chess Analysis

First we establish a numbering for all the squares.
A square can be identified by a (row, column) pair.
The rows are numbered from top to bottom, starting at 0.
The columns are numbered from left to right, also starting at 0.

Let $D^m_{(r,c)}$ represent the square $(r+m, c+2m)$, where r, c and m can come from the set $\mathbb{N} \cup \{0\}$.

Now that this has been established we can proceed to prove some claims.
<u>Claim:</u>

$$\mathcal{N}(D^m_{(0,i)}) = m, \text{ for any } i \geq 0$$

<u>Proof:</u>
We will use strong induction.

Base Cases:
$\mathcal{N}(D^0_{(0,i)}) = 0$
$\mathcal{N}(D^1_{(0,i)}) = 1$
The values for the base cases can be read off the figure on the previous page.

Induction Hypothesis:
Suppose for some $m \in \mathbb{N}$, the claim is true for all $k < m$.

Induction Step:
We want to prove that the claim is true for $m+1$.

Consider the square $D^{m+1}_{(0,i)}$, and notice that we can reach any square of the form $D^i_{(0,i)}$ where $0 \leq i \leq m$ from this square using a single-up double-left move. So by the I.H., we can reach any position with nimber i where $0 \leq i \leq m$ from the square using a single-up double-left move.

Now consider an arbitrary square reachable from $D^{m+1}_{(0,i)}$ with a double-up single-left move. We can only make *less than* $m+1$ single-up double-left or *less than* $m+1$ double-up single-left moves from such a square to reach the sides of the square, so it is just not possible for such a square to have a nimber greater than or equal to $m+1$.

Thus, the set of nimbers of all the positions reachable from $D^{m+1}_{(0,i)}$ is just $\{0, 1, 2, \ldots, m\}$. By the Sprague-Grundy Theorem, $\mathcal{N}(D^{m+1}_{(0,i)}) = m+1$ because $m+1 = \text{mex}(\{0, 1, 2, \ldots, m\})$.

Thus the claim is true for $m+1$. ∎

Claim:

$$\mathcal{N}(D^m_{(2,1)}) = \begin{cases} (m+1), & \text{if } m \equiv 0 \pmod 2 \\ (m-1), & \text{if } m \equiv 1 \pmod 2 \end{cases}$$

Proof:
We will use a variant of strong induction.

Base Cases:
$\mathcal{N}(D^0_{(2,1)}) = 1$
$\mathcal{N}(D^1_{(2,1)}) = 0$
$\mathcal{N}(D^2_{(2,1)}) = 3$
$\mathcal{N}(D^3_{(2,1)}) = 2$
The values for the base cases can be read off the figure on the previous page.

Induction Hypothesis:
Suppose for some $m \in \mathbb{N}$, $m \equiv 1 \pmod 2$ the claim is true for all $k < m$.

Induction Step:
We want to prove that the claim is true for $m+1$ and $m+2$.

We need to prove that $\mathcal{N}(D^{m+1}_{(2,1)}) = m+2$.
From $D^{m+1}_{(2,1)}$, we can reach any square of the form $D^{2j}_{(2,1)}$ where $0 \leq j \leq \frac{m+1}{2} - 1$ using a single-up double-left move.
By the induction hypothesis, this means that from $D^{m+1}_{(2,1)}$ we can reach squares with nimbers in the set $\{2j+1 : 0 \leq j \leq \frac{m-1}{2}\} = \{o : o \text{ is an odd number less than } m+1\}$.
From $D^{m+1}_{(2,1)}$, we can also reach any square of the form $D^{2j+1}_{(2,1)}$ where $0 \leq j \leq \frac{m-1}{2}$ using a single-up double-left move.
this means that from $D^{m+1}_{(2,1)}$ we can reach squares with nimbers in the set $\{2j : 0 \leq j \leq \frac{m-1}{2}\} = \{e : e \text{ is an even number less than } m+1\}$.
It follows that from $D^{m+1}_{(2,1)}$, we can definitely reach game states with nimbers in the set $\{0, 1, \ldots, m\}$.
Now consider an arbitrary square reachable from $D^{m+1}_{(2,1)}$ with a double-up single-left move. We can only make *less than* $m+1$ single-up double-left or *less than* $m+1$ double-up single-left moves from such a square to reach the sides of the square, so it is just not possible for such a square to have a nimber greater than $m+1$.
But notice that when we start out at $D^{m+1}_{(2,1)}$ and then move two squares up and one square to the left we get to the square $D^{m+1}_{(0,0)}$, whose nimber — as we proved earlier — is $m+1$.
No other squares reachable from $D^{m+1}_{(2,1)}$ matter to us as they have lower nimbers.
We have shown that from this square we can get to game states whose nimbers cover the set $\{0, 1, \ldots, m+1\}$.
By the Sprague-Grundy Theorem, $\mathcal{N}(D^{m+1}_{(2,1)}) = \text{mex}(\{0, 1, \ldots, m+1\}) = m+2$.
This means that the claim holds for $m+2$.

Now consider the square $D^{m+2}_{(2,1)}$.
We want to show $\mathcal{N}(D^{m+2}_{(2,1)}) = m+1$.
We already know that using single-up double-left moves from $D^{m+2}_{(2,1)}$, the nimbers of the game states that we can reach cover the set $(\{0, 1, \ldots, m+2\} \setminus \{m+1\})$.
Notice that $D^{m+2}_{(2,1)} = (m+4, 2m+5)$.
If we move two squares up and one square left from here we reach $(m+2, 2m+4) = D^{m+2}_{(0,0)}$ which has nimber $m+2$.
If we move another two squares up and one square left we get to $(m, 2m+3) = D^m_{(0,3)}$, which has nimber m.
Moving a further two squares up and one square left takes us to $(m-2, 2m+2)$. Note that from here the number of possible single-up double-left moves is $m-2$ and the number of double-up single-left moves is $\frac{m-2}{2}$, so this square has a nimber that is less than or equal to $m-1$.
Thus all the other moves of the double-up single-left variety are irrelevant to us.
It follows that the set of the nimbers of all the squares that we can reach from $D^{m+2}_{(2,1)}$ is $(\{0, 1, \ldots, m+2\} \setminus \{m+1\})$. The mex of this set is $m+1$, so by the Sprague-Grundy Theorem, $\mathcal{N}(D^{m+2}_{(2,1)}) = m+1$.

We have proved that the claim holds for $m+1$ and $m+2$. ■

Claim:

$$\mathcal{N}(D^m_{(3,0)}) = \begin{cases} m, & \text{if } m \equiv 0 \pmod 4 \\ m+1, & \text{if } m \equiv 1,2 \pmod 4 \\ m-2, & \text{if } m \equiv 3 \pmod 4 \end{cases}$$

Proof:
We will use a variant of strong induction.

Base Cases:
$\mathcal{N}(D^0_{(3,0)}) = 0$
$\mathcal{N}(D^1_{(3,0)}) = 2$
$\mathcal{N}(D^2_{(3,0)}) = 3$
$\mathcal{N}(D^3_{(3,0)}) = 1$
$\mathcal{N}(D^4_{(3,0)}) = 4$
$\mathcal{N}(D^5_{(3,0)}) = 6$
$\mathcal{N}(D^6_{(3,0)}) = 7$
$\mathcal{N}(D^7_{(3,0)}) = 5$
The values for the base cases can be read off the figure on the previous page.

Induction Hypothesis:
Suppose for some $m \in \mathbb{N}$, $m \equiv 3 \pmod 4$ the claim is true for all $k < m$.

Induction Step:
We want to prove that the claim is true for $m+1$, $m+2$, $m+3$ and $m+4$.

First consider $D^{m+1}_{(3,0)}$.
From this square, we can get to any square of the form $D^{4j}_{(3,0)}$ where $0 \leq j \leq \frac{m+1}{4} - 1$ using one single-up double-left move.
By the I.H., the nimbers of these squares make up the set $\{4j : 0 \leq j \leq \frac{m+1}{4} - 1\} = \{k : k \equiv 0 \pmod 4 \text{ and } k < m+1\}$.
From $D^{m+1}_{(3,0)}$, we can also get to any square of the form $D^{4j+1}_{(3,0)}$ where $0 \leq j \leq \frac{m+1}{4} - 1$ using one single-up double-left move.
By the I.H., the nimbers of these squares make up the set $\{4j+2 : 0 \leq j \leq \frac{m+1}{4} - 1\} = \{k : k \equiv 2 \pmod 4 \text{ and } k < m+1\}$.
We can also get to any square of the form $D^{4j+2}_{(3,0)}$ where $0 \leq j \leq \frac{m+1}{4} - 1$ using one single-up double-left move.
By the I.H., the nimbers of these squares make up the set $\{4j+3 : 0 \leq j \leq \frac{m+1}{4} - 1\} = \{k : k \equiv 3 \pmod 4 \text{ and } k < m+1\}$.
Finally, we can get to any square of the form $D^{4j+3}_{(3,0)}$ where $0 \leq j \leq \frac{m+1}{4} - 1$ using one single-up double-left move.
By the I.H., the nimbers of these squares make up the set $\{4j+1 : 0 \leq j \leq \frac{m+1}{4} - 1\} = \{k : k \equiv 1 \pmod 4 \text{ and } k < m+1\}$.
So, finding the union of all these sets, we gather that using a single-up double-left move, all the nimbers we can reach make up the set $\{0, 1, \ldots, m\}$.
What if we use a double-up single-left move from $D^{m+1}_{(3,0)}$ though?
Moving two squares up and one square to the left gets us to $D^m_{(2,1)}$ where $m \equiv 3 \pmod 4 \Rightarrow m \equiv 1 \pmod 2$, $\mathcal{N}(D^m_{(2,1)}) = m-1$ according to our previous proof.
From here, moving two squares up and one square to the left gets us to $D^m_{(0,0)}$ and we know $\mathcal{N}(D^m_{(0,0)}) = m$.
Finally, moving another two squares up and one square to the left gets us to $D^{m-2}_{(0,3)}$ and we know that the nimber here or any square to the top-left of this one cannot be $m+1$ or more because there is not enough space to make m disinct moves of either the single-up double-left variety or the double-up single-left variety.
It follows that the set of all the nimbers of the children of $D^{m+1}_{(0,3)}$ is $\{0, 1, \ldots, m\}$. The mex of this set is $m+1$ and by the Sprague-Grundy theorem $\mathcal{N}(D^{m+1}_{(0,3)}) = m+1$.
This proves that the claim holds for $m+1$.

Now consider $D^{m+2}_{(3,0)}$.
We are already aware that the set of nimbers of all the child states that can be reached from this square using single-up double-left moves is $\{0, 1, \ldots, m+1\}$.
So let us move two squares up and one square left from here to reach $D^{m+1}_{(2,1)}$, where $m+1 \equiv 0 \pmod 2$. This means $\mathcal{N}(D^{m+1}_{(2,1)}) = m+2$.
Moving two more squares up and one square left gets us to $D^{m+1}_{(0,0)}$ which has nimber $m+1$.
Finally moving a further two squares up and one square left gets us to $D^{m-1}_{(0,3)}$ which cannot have a nimber greater than or equal to $m+1$.
So the set of nimbers of the squares reachable from $D^{m+2}_{(3,0)}$ in one move is $\{0, 1, \ldots, m+2\}$ whose mex is $m+3$.
Thus $\mathcal{N}(D^{m+2}_{(3,0)}) = m+3$. The claim holds for $m+2$ as well.

Now consider $D^{m+3}_{(3,0)}$.
We are already aware that the set of nimbers of all the child states that can be reached from this square using single-up double-left moves is $\{0, 1, \ldots, m+1\} \cup \{m+3\}$.
So let us move two squares up and one square left from here to reach $D^{m+2}_{(2,1)}$, where $m + 2 \equiv 1 \pmod 2$. This means $\mathcal{N}(D^{m+2}_{(2,1)}) = m + 1$.
Moving two more squares up and one square left gets us to $D^{m+2}_{(0,0)}$ which has nimber $m + 2$.
Finally moving a further two squares up and one square left gets us to $D^{m}_{(0,3)}$ which cannot have a nimber greater than or equal to $m + 1$.
So the set of nimbers of the squares reachable from $D^{m+3}_{(3,0)}$ in one move is $\{0, 1, \ldots, m+3\}$ whose mex is $m + 4$.
Thus $\mathcal{N}(D^{m+3}_{(3,0)}) = m + 4$. So, the claim holds for $m + 3$ as well.

Finally consider $D^{m+4}_{(3,0)}$.
We are aware that the set of nimbers of all the child states that can be reached from this square using single-up double-left moves is $\{0, 1, \ldots, m+1\} \cup \{m+3, m+4\}$.
So let us move two squares up and one square left from here to reach $D^{m+3}_{(2,1)}$, where $m + 3 \equiv 0 \pmod 2$. This means $\mathcal{N}(D^{m+3}_{(2,1)}) = m + 4$.
Moving two more squares up and one square left gets us to $D^{m+3}_{(0,0)}$ which has nimber $m + 3$.
Moving a further two squares up and one square left gets us to $D^{m+1}_{(0,3)}$ which cannot have a nimber greater than or equal to $m + 2$.
So the set of nimbers of the squares reachable from $D^{m+3}_{(3,0)}$ in one move is $\{0, 1, \ldots, m+1\} \cup \{m+3, m+4\}$ whose mex is $m + 2$.
Thus $\mathcal{N}(D^{m+4}_{(3,0)}) = m + 2$. So, the claim holds for $m + 4$ as well.

We proved the claim for $m + 1$, $m + 2$, $m + 3$ and $m + 4$. This allows us to conclude, by strong induction, that the proposed formula holds for all $m \in \mathbb{N}$. ■

Chess Possibilities

The observation that we can make by looking at the nimbers (using a program to calculate nimbers) is that the values of $D^{m+1}_{(r,c)} - D^{m}_{(r,c)}$ for $m \geq 0$ may initially appear to be unstructured and then settle into a cycle, repeating periodically. Also, there is a different cycle for each new initial (r, c) used as a starting point for $D^{m}_{(r,c)}$.

Some cycles that have been observed (but not proved) follow. They have been matched up with the appropriate starting squares in (row, column) notation. Keep in mind that some of sequences start with seemingly unpredictable nimbers and those parts have been disregarded.

```
(7 , 7):  5, 1, 1,-3
(8 , 8):  5, 1, 1,-3, 4, 2, 1, 1,-3
(9 , 9):  2, 1,-2, 3
(10,10):  6, 1, 1, 1,-4, 6, 1, 1, 1,-4, 5, 2, 1, 1, 1,-4
(11,11): -1,-2, 5, 1, 2,-4, 6, 1,-4, 6,-4, 5, 1,-3, 6,-1,
          2,-3, 4, 3, 1, 1,-4, 1, 4, 2, 2
(12,12):  1, 1,-5, 7, 1, 1
(13,13):  7, 1,-5, 6, 1,-4, 7, 1, 1, 1,-5, 1, 5, 3, 1, 1,
          1,-5, 1, 6, 1, 2, 1,-5, 7, 1,-5, 7,-1,-3, 6, 2,
         -1, 3,-5, 6, 1, 2,-5, 7,-1, 2,-4, 7,-1, 3, 1,-5
(14,14):  1, 1, 1,-4, 6,-1, 2, 2,-1, 3, 1, 1, 1,-4, 6, 1,
          1, 1,-4, 5, 1, 2, 1, 1, 1,-4, 6
(15,15):  1, 1,-6, 1, 8, 1, 1, 1, 1,-6, 1, 8, 1, 1, 1, 1,
         -6, 1, 8, 1, 1, 1, 1,-6, 1, 7, 2, 1, 1, 1,-6, 7,
         -5, 8, 1, 1
```

The idea is that these can lead us to new formulae worth proving (like the ones proved in the analysis section).